Asymptotic Performance of Complex M-estimators for Multivariate Location and Scatter Estimation

Bruno Mériaux, *Student Member, IEEE,* Chengfang Ren, *Member, IEEE,*
Mohammed Nabil El Korso, Arnaud Breloy, *Member, IEEE,* and Philippe Forster

***Abstract*—The joint estimation of means and scatter matrices is often a core problem in multivariate analysis. In order to overcome robustness issues, such as outliers from Gaussian assumption, M-estimators are now preferred to the traditional sample mean and sample covariance matrix. These estimators are well established and studied in the real case since the seventies. Their extension to the complex case has drawn recent interest. In this letter, we derive the asymptotic performance of complex M-estimators for multivariate location and scatter matrix estimation.**

***Keywords*—Complex observations, Robust estimation of multivariate location and scatter, Complex Elliptically Symmetric distributions.**

I. Introduction

SEVERAL classical methods in multivariate analysis require the estimation of means and scatter matrices from collected observations [1]–[5]. In pratice, the Sample Mean and the Sample Covariance Matrix are classically used in such procedures. Indeed, they coincide with the Maximum Likelihood Estimators (MLE) for multivariate Gaussian data. However, they are neither robust to deviation from Gaussianity assumption nor to the presence of outliers, which could lead to a dramatic performance loss. To overcome these problems, several approaches have been proposed in the literature [6], [7] improving the estimator's behavior under contamination through diverse criteria such as the breakdown point, the contamination bias, the finite-sample efficiency while preserving the computationally feasibility for high dimension. Among the most frequently used robust affine equivariant estimators [8]–[17], we focus on the M-estimators, which have been first introduced within the framework of elliptical distributions [18]. The latters encompass a large number of classical distributions as for instance the Gaussian one but also non-Gaussian heavy-tailed distributions such as the t-, K- and W-distributions [19]. One of the main interests of the real M-estimators is to possess, under mild conditions, good asymptotic properties, namely weak consistency and asymptotic normality over the whole class of elliptical distributions [8], [20]–[22]. Their extension to the complex case has drawn recent interest [23], notably with the class of Complex Elliptically Symmetric (CES) distributions [24]. In the context of known mean, their asymptotic properties are established in [25], [26]. In this letter, we extend this result for multivariate location and scatter matrix M-estimates under CES distributed data. The achieved outcome is analogous to its real-counterpart [21] and may be used for deriving the performance of adaptive processes in non-zero mean non Gaussian distributed observations [1].

B. Mériaux and C. Ren are with SONDRA, CentraleSupélec, France. M.N. El Korso and A. Breloy are with LEME, Paris-Nanterre University, France. P. Forster is with SATIE, Paris-Nanterre University, France.

This work is financed by the Direction Générale de l'Armement as well as the ANR ASTRID referenced ANR-17-ASTR-0015.

In the following, the notation $\overset{d}{=}$, $\overset{d}{\to}$ and $\overset{\mathbb{P}}{\to}$ indicate respectively equality in distribution, convergence in law and in probability. The symbol $\perp$ refers to statistical independence. The operator $\text{vec}(\mathbf{A})$ stacks all columns of $\mathbf{A}$, designated by $(\mathbf{a}_1, \dots, \mathbf{a}_i, \dots)$ into a vector. The operator $\otimes$ refers to Kronecker product. The notation $\mathcal{G}\mathbb{C}\mathcal{N}(\mathbf{0}, \mathbf{\Sigma}, \mathbf{\Omega})$ refers to the zero mean non-circular complex Gaussian distribution, where $\mathbf{\Sigma}$ (respectively $\mathbf{\Omega}$) denotes the covariance matrix (respectively pseudo-covariance matrix) [27], [28].

II. Problem setup

Let $\mathbf{Z}_N \triangleq (\mathbf{z}_1, \dots, \mathbf{z}_N)$ be N i.i.d samples of m-dimensional vectors, following a complex elliptical distribution, which is denoted by $\mathbf{z}_n \sim \mathbb{C}\mathcal{ES}_m(\mathbf{t}_e, \mathbf{\Lambda}, g_{\mathbf{z}})$ and whose p.d.f. is proportional to

$$p_{\mathbf{z}}(\mathbf{z}_n; \mathbf{t}_e, \mathbf{\Lambda}, g_{\mathbf{z}}) \propto \det(\mathbf{\Lambda})^{-1} g_{\mathbf{z}}\left((\mathbf{z}_n - \mathbf{t}_e)^H \mathbf{\Lambda}^{-1}(\mathbf{z}_n - \mathbf{t}_e)\right). \quad (1)$$

The vector $\mathbf{t}_e$ is the location parameter, $\mathbf{\Lambda}$ denotes the scatter matrix and the function $g_{\mathbf{z}}$ is the density generator [29]. We aim to estimate jointly the location parameter and the scatter matrix from the observations. The complex joint M-estimators $\left(\hat{\mathbf{t}}_N, \widehat{\mathbf{M}}_N\right)$ are solutions of the system $\mathcal{S}ys_N(\mathbf{Z}_N, u_1, u_2)$ [23]:

$$\begin{cases} \dfrac{1}{N}\displaystyle\sum_{n=1}^{N} u_1\left(d(\mathbf{z}_n, \mathbf{t}; \mathbf{M})\right)(\mathbf{z}_n - \mathbf{t}) = \mathbf{0} & (2) \\ \underbrace{\dfrac{1}{N}\displaystyle\sum_{n=1}^{N} u_2\left(d^2(\mathbf{z}_n, \mathbf{t}; \mathbf{M})\right)(\mathbf{z}_n - \mathbf{t})(\mathbf{z}_n - \mathbf{t})^H}_{\mathcal{H}(\mathbf{Z}_N, \mathbf{t}, \mathbf{M})} = \mathbf{M} & (3) \end{cases}$$

with $d^2(\mathbf{z}, \mathbf{t}; \mathbf{M}) = (\mathbf{z} - \mathbf{t})^H \mathbf{M}^{-1}(\mathbf{z} - \mathbf{t})$. Let us consider $\left(\mathbf{t}_e, \mathbf{M}_e\right)$ a solution related to the system $\mathcal{S}ys_\infty(\mathbf{z}_1, u_1, u_2)$:

$$\begin{cases} \mathbb{E}\left[u_1\left(d(\mathbf{z}_1, \mathbf{t}; \mathbf{M})\right)(\mathbf{z}_1 - \mathbf{t})\right] = \mathbf{0} & (4) \\ \mathbb{E}\left[u_2\left(d^2(\mathbf{z}_1, \mathbf{t}; \mathbf{M})\right)(\mathbf{z}_1 - \mathbf{t})(\mathbf{z}_1 - \mathbf{t})^H\right] \triangleq \mathcal{H}_\infty(\mathbf{t}, \mathbf{M}) = \mathbf{M} & (5) \end{cases}$$

The functions $u_1(\cdot)$ and $u_2(\cdot)$ verify the conditions given in [8] for the real case (for the special case, where $u_1(s) = u_2(s^2)$, [30] provides more general conditions). The proofs of Lemmas 1 and 2 and Theorems 1-3 of [8], addressing the existence and uniqueness of M-estimates, are transposable to the complex field, by following the same methodology as in [8]. The derivations of these proofs require the same conditions on $u_1(\cdot)$ and $u_2(\cdot)$ as the ones needed in the real case. Thus, this ensures the existence of $\left(\hat{\mathbf{t}}_N, \widehat{\mathbf{M}}_N\right)$ and $\left(\mathbf{t}_e, \mathbf{M}_e\right)$ as well as the uniqueness of $\left(\mathbf{t}_e, \mathbf{M}_e\right)$. In this letter, we derive the statistical performance of the complex joint M-estimators $\left(\hat{\mathbf{t}}_N, \widehat{\mathbf{M}}_N\right)$, namely consistency and asymptotic distribution.

III. Consistency of the joint M-estimator

Let $\left(\hat{\mathbf{t}}_N, \widehat{\mathbf{M}}_N\right)$ be a solution of $\mathcal{S}ys_N\left(\mathbf{Z}_N, u_1, u_2\right)$ and $\left(\mathbf{t}_e, \mathbf{M}_e\right)$ be the solution of the system $\mathcal{S}ys_\infty\left(\mathbf{z}_1, u_1, u_2\right)$.

Theorem III.1. The complex joint M-estimators are consistent, i.e.

$$\left(\hat{\mathbf{t}}_N, \widehat{\mathbf{M}}_N\right) \xrightarrow{\mathbb{P}} \left(\mathbf{t}_e, \mathbf{M}_e\right). \tag{6}$$

with $\mathbf{M}_e = \sigma^{-1}\boldsymbol{\Lambda}$ in which σ is the solution of $\mathbb{E}\left[\psi_2\left(\sigma|\boldsymbol{\zeta}|^2\right)\right] = m$, for $\boldsymbol{\zeta} \sim \mathbb{C}\mathcal{ES}_m\left(\mathbf{0}_m, \mathbf{I}_m, g_{\mathbf{z}}\right)$ and $\psi_2(s) = s u_2(s)$.

Proof: First, let us define $\boldsymbol{\theta}^T = \left[\mathbf{t}^T, \mathrm{vec}\left(\mathbf{M}\right)^T\right]$ and the function $\boldsymbol{\Psi}_N\left(\boldsymbol{\theta}\right) = \begin{bmatrix} \boldsymbol{\Psi}_{1,N}\left(\boldsymbol{\theta}\right) = \frac{1}{N}\sum\limits_{n=1}^{N} u_1\left(d\left(\mathbf{z}_n, \mathbf{t}; \mathbf{M}\right)\right)\left(\mathbf{z}_n - \mathbf{t}\right) \\ \boldsymbol{\Psi}_{2,N}\left(\boldsymbol{\theta}\right) = \mathrm{vec}\left(\mathcal{H}\left(\mathbf{Z}_N, \mathbf{t}, \mathbf{M}\right) - \mathbf{M}\right)\end{bmatrix}$.

The Strong Law of Large Numbers (SLLN) gives

$$\forall \boldsymbol{\theta} \in \boldsymbol{\Theta} \quad \boldsymbol{\Psi}_N\left(\boldsymbol{\theta}\right) \xrightarrow{\mathbb{P}} \boldsymbol{\Psi}\left(\boldsymbol{\theta}\right) \tag{7}$$

with $\forall \boldsymbol{\theta} \in \boldsymbol{\Theta}$, $\boldsymbol{\Psi}\left(\boldsymbol{\theta}\right) = \begin{bmatrix} \boldsymbol{\Psi}_1\left(\boldsymbol{\theta}\right) = \mathbb{E}\left[u_1\left(d\left(\mathbf{z}_1, \mathbf{t}; \mathbf{M}\right)\right)\left(\mathbf{z}_1 - \mathbf{t}\right)\right] \\ \boldsymbol{\Psi}_2\left(\boldsymbol{\theta}\right) = \mathrm{vec}\left(\mathcal{H}_\infty\left(\mathbf{t}, \mathbf{M}\right) - \mathbf{M}\right)\end{bmatrix}$.

According to the Theorem 5.9 [31, Chap. 5] and uniqueness of solution, we can show that any $\widehat{\boldsymbol{\theta}}_N^T = \left[\hat{\mathbf{t}}_N^T, \mathrm{vec}\left(\widehat{\mathbf{M}}_N\right)^T\right]$ solution of $\boldsymbol{\Psi}_N\left(\widehat{\boldsymbol{\theta}}_N\right) = \mathbf{0}$ converges in probability to $\boldsymbol{\theta}_e^T = \left[\mathbf{t}_e^T, \mathrm{vec}\left(\mathbf{M}_e\right)^T\right]$ solution of $\boldsymbol{\Psi}\left(\boldsymbol{\theta}_e\right) = \mathbf{0}$, yielding the intended outcome. Furthermore, the matrix $\mathbf{M}_e$ is proportional to $\boldsymbol{\Lambda}$ through a scale factor σ^{-1} [32, Chap. 6]. Multiplying (5) by $\mathbf{M}^{-1}$ and taking the trace yields $\mathbb{E}\left[\psi_2\left(\sigma|\boldsymbol{\zeta}|^2\right)\right] = m$ of which σ is the solution. ∎

IV. Asymptotic distribution of the joint M-estimators

A. Main theorem

We consider $\left(\hat{\mathbf{t}}_N, \widehat{\mathbf{M}}_N\right)$ a solution of $\mathcal{S}ys_N\left(\mathbf{Z}_N, u_1, u_2\right)$ as well as $\left(\mathbf{t}_e, \mathbf{M}_e\right)$ the solution of $\mathcal{S}ys_\infty\left(\mathbf{z}_1, u_1, u_2\right)$.

Theorem IV.1. Assuming that $s\psi_i^{'}(s)$ $(i = 1, 2)$ are bounded and $\mathbb{E}\left[\psi_1^{'}\left(\sqrt{\sigma}|\boldsymbol{\zeta}|\right)\right] > 0$, the asymptotic distribution of $\left(\hat{\mathbf{t}}_N, \widehat{\mathbf{M}}_N\right)$ is given by

$$\sqrt{N}\left(\left(\hat{\mathbf{t}}_N - \mathbf{t}_e, \mathrm{vec}\left(\widehat{\mathbf{M}}_N - \mathbf{M}_e\right)\right)\right) \xrightarrow{d} \left(\mathbf{a}, \mathbf{b}\right) \text{ with } \mathbf{a} \perp \mathbf{b} \text{ and}$$

$$\mathbf{a} \sim \mathbb{C}\mathcal{N}\left(\mathbf{0}, \boldsymbol{\Sigma}_t = \frac{\alpha}{\beta^2}\mathbf{M}_e\right),\ \mathbf{b} \sim \mathcal{G}\mathbb{C}\mathcal{N}\left(\mathbf{0}, \boldsymbol{\Sigma}_M, \boldsymbol{\Omega}_M = \boldsymbol{\Sigma}_M\mathbf{K}_m\right)$$

where $\mathbf{K}_m$ is the commutation matrix satisfying $\mathbf{K}_m\mathrm{vec}\left(\mathbf{A}\right) = \mathrm{vec}\left(\mathbf{A}^T\right)$ [33] and $\boldsymbol{\Sigma}_M$ is obtained by

$$\boldsymbol{\Sigma}_M = \sigma_1\mathbf{M}_e^T \otimes \mathbf{M}_e + \sigma_2\mathrm{vec}\left(\mathbf{M}_e\right)\mathrm{vec}\left(\mathbf{M}_e\right)^H, \tag{8}$$

in which

$$\begin{cases} \alpha = m^{-1}\mathbb{E}\left[\psi_1^2(\sqrt{\sigma}|\boldsymbol{\zeta}|)\right] \\ \beta = \mathbb{E}\left[\left(1 - (2m)^{-1}\right)u_1(\sqrt{\sigma}|\boldsymbol{\zeta}|) + (2m)^{-1}\psi_1^{'}\left(\sqrt{\sigma}|\boldsymbol{\zeta}|\right)\right] \\ \sigma_1 = \dfrac{a_1(m+1)^2}{(a_2+m)^2}, \sigma_2 = a_2^{-2}\left[(a_1 - 1) - a_1(a_2 - 1)\dfrac{m + (m+2)a_2}{(a_2+m)^2}\right] \\ a_1 = \dfrac{\mathbb{E}\left[\psi_2^2\left(\sigma|\boldsymbol{\zeta}|^2\right)\right]}{m(m+1)},\ a_2 = \dfrac{\mathbb{E}\left[\sigma|\boldsymbol{\zeta}|^2\psi_2^{'}\left(\sigma|\boldsymbol{\zeta}|^2\right)\right]}{m}\end{cases}$$

with σ solution of $\mathbb{E}\left[\psi_2\left(\sigma|\boldsymbol{\zeta}|^2\right)\right] = m$ in which $\boldsymbol{\zeta} \sim \mathbb{C}\mathcal{ES}_m\left(\mathbf{0}_m, \mathbf{I}_m, g_{\mathbf{z}}\right)$.

B. Proof of Theorem IV.1

The starting point of the proof is to map the complex joint M-estimators $\left(\hat{\mathbf{t}}_N, \widehat{\mathbf{M}}_N\right)$ into real-ones, then to study the asymptotic behavior of the latter, and finally to relate the latter to the asymptotic distribution of the complex joint M-estimators $\left(\hat{\mathbf{t}}_N, \widehat{\mathbf{M}}_N\right)$.

1) Complex vector space isomorphism: Let us first introduce functions $h : \mathbb{C}^m \to \mathbb{R}^p$ and $f : \mathbb{C}^{m\times m} \to \mathbb{R}^{p\times p}$ with $p = 2m$ defined by $h\left(\mathbf{a}\right) = \left(\Re\left(\mathbf{a}\right)^T, \Im\left(\mathbf{a}\right)^T\right)^T$ and

$$f\left(\mathbf{A}\right) = \frac{1}{2}\begin{pmatrix}\Re\left(\mathbf{A}\right) & -\Im\left(\mathbf{A}\right) \\ \Im\left(\mathbf{A}\right) & \Re\left(\mathbf{A}\right)\end{pmatrix}$$

In addition, let $\boldsymbol{\mathcal{P}} \in \mathbb{R}^{p\times p}$ be the matrix $\boldsymbol{\mathcal{P}} = \begin{pmatrix}\mathbf{0}_{m\times m} & -\mathbf{I}_m \\ \mathbf{I}_m & \mathbf{0}_{m\times m}\end{pmatrix}$. Some useful properties of the previous functions are given in [25]. Furthermore, we set $\hat{\mathbf{t}}_N^{\mathbb{R}} = h\left(\hat{\mathbf{t}}_N\right)$, $\widehat{\mathbf{M}}_N^{\mathbb{R}} = f\left(\widehat{\mathbf{M}}_N\right)$, $\mathbf{M}_{\mathbb{R}} = f\left(\mathbf{M}_e\right)$ and $\mathbf{t}_{\mathbb{R}} = h\left(\mathbf{t}_e\right)$. In addition, let us define $\mathbf{u}_n = h\left(\mathbf{z}_n\right) \sim \mathcal{ES}_p\left(\mathbf{t}_e^u, \boldsymbol{\Lambda}_{\mathbb{R}}, g_{\mathbf{z}}\right)$ and $\mathbf{v}_n = \boldsymbol{\mathcal{P}}\mathbf{u}_n \sim \mathcal{ES}_p\left(\mathbf{t}_e^v, \boldsymbol{\Lambda}_{\mathbb{R}}, g_{\mathbf{z}}\right)$ for $\mathbf{z}_n \sim \mathbb{C}\mathcal{ES}_m\left(\mathbf{t}_e, \boldsymbol{\Lambda}, g_{\mathbf{z}}\right)$, where $\mathbf{t}_e^u = h\left(\mathbf{t}_e\right)$, $\mathbf{t}_e^v = \boldsymbol{\mathcal{P}}\mathbf{t}_e^u$ and $\boldsymbol{\Lambda}_{\mathbb{R}} = f\left(\boldsymbol{\Lambda}\right)$. The notation $\mathcal{ES}$ refers to real elliptical distributions [18]. Moreover, there exist another relation between the vectors $\mathbf{z}_n$, $\mathbf{u}_n$ and $\mathbf{v}_n$, for any Hermitian matrix, $\mathbf{A} \in \mathbb{C}^{m\times m}$ [25]:

$$2\mathbf{z}_n^H\mathbf{A}^{-1}\mathbf{z}_n = \mathbf{u}_n^T f\left(\mathbf{A}\right)^{-1}\mathbf{u}_n = \mathbf{v}_n^T f\left(\mathbf{A}\right)^{-1}\mathbf{v}_n \tag{9}$$

Let us apply the function $f(\cdot)$ to the equation (3) (respectively $h(\cdot)$ to (2)), we obtain

$$\widehat{\mathbf{M}}_N^{\mathbb{R}} = \frac{1}{2N}\sum_{n=1}^{N} u_{2,\mathbb{R}}\left(d^2\left(\mathbf{u}_n, \hat{\mathbf{t}}_N^{\mathbb{R}}; \widehat{\mathbf{M}}_N^{\mathbb{R}}\right)\right)\left(\mathbf{u}_n - \hat{\mathbf{t}}_N^{\mathbb{R}}\right)\left(\mathbf{u}_n - \hat{\mathbf{t}}_N^{\mathbb{R}}\right)^T \tag{10}$$

$$+ \frac{1}{2N}\sum_{n=1}^{N} u_{2,\mathbb{R}}\left(d^2\left(\mathbf{v}_n, \boldsymbol{\mathcal{P}}\hat{\mathbf{t}}_N^{\mathbb{R}}; \widehat{\mathbf{M}}_N^{\mathbb{R}}\right)\right)\left(\mathbf{v}_n - \boldsymbol{\mathcal{P}}\hat{\mathbf{t}}_N^{\mathbb{R}}\right)\left(\mathbf{v}_n - \boldsymbol{\mathcal{P}}\hat{\mathbf{t}}_N^{\mathbb{R}}\right)^T$$

$$\text{and } \frac{1}{N}\sum_{n=1}^{N} u_{1,\mathbb{R}}\left(d\left(\mathbf{u}_n, \hat{\mathbf{t}}_N^{\mathbb{R}}; \widehat{\mathbf{M}}_N^{\mathbb{R}}\right)\right)\left(\mathbf{u}_n - \hat{\mathbf{t}}_N^{\mathbb{R}}\right) = \mathbf{0} \tag{11}$$

where $u_{2,\mathbb{R}}(s) = u_2\left(2^{-1}s\right)$ and $u_{1,\mathbb{R}}(s) = u_1\left(2^{-1/2}s\right)$ according to (9). Let $\psi_{i,\mathbb{R}}(\cdot)$ be the functions related to $u_{i,\mathbb{R}}(\cdot)$ by $\psi_{i,\mathbb{R}}(s) = s u_{i,\mathbb{R}}(s)$, $i = 1, 2$. Finally, we introduce the two following real joint M-estimators $\left(\hat{\mathbf{t}}_N^u, \widehat{\mathbf{M}}_N^u\right)$ and $\left(\hat{\mathbf{t}}_N^v, \widehat{\mathbf{M}}_N^v\right)$ respectively solution of $\mathcal{S}ys_N\left(\mathbf{U}_N, u_{1,\mathbb{R}}, u_{2,\mathbb{R}}\right)$ and $\mathcal{S}ys_N\left(\mathbf{V}_N, u_{1,\mathbb{R}}, u_{2,\mathbb{R}}\right)$. From the results in the real case on the consistency [8], [21], we obtain

$$\begin{cases} \left(\hat{\mathbf{t}}_N^u, \widehat{\mathbf{M}}_N^u\right) \xrightarrow{\mathbb{P}} \left(\mathbf{t}_u, \mathbf{M}_u\right) = \left(\mathbf{t}_e^u, \sigma_{\mathbb{R}}^{-1}\mathbf{\Lambda}_{\mathbb{R}}\right) \\ \left(\hat{\mathbf{t}}_N^v, \widehat{\mathbf{M}}_N^v\right) \xrightarrow{\mathbb{P}} \left(\mathbf{t}_v, \mathbf{M}_v\right) = \left(\mathbf{t}_e^v, \sigma_{\mathbb{R}}^{-1}\mathbf{\Lambda}_{\mathbb{R}}\right) \end{cases} \tag{12}$$

in which $(\mathbf{t}_u, \mathbf{M}_u)$ and $(\mathbf{t}_v, \mathbf{M}_v)$ are solutions of $\mathcal{S}ys_\infty\left(\mathbf{u}_1, u_{1,\mathbb{R}}, u_{2,\mathbb{R}}\right)$ and $\mathcal{S}ys_\infty\left(\mathbf{v}_1, u_{1,\mathbb{R}}, u_{2,\mathbb{R}}\right)$ and $\sigma_{\mathbb{R}}$ is the solution of $\mathbb{E}\left[\psi_{2,\mathbb{R}}\left(\sigma_{\mathbb{R}}|\mathbf{u}|^2\right)\right] = p = 2m$, $\mathbf{u} \sim \mathcal{ES}_p\left(\mathbf{0}, \mathbf{I}_p, g_{\mathbf{z}}\right)$. Thus, we have $\mathbf{M}_u = \sigma_{\mathbb{R}}^{-1}\mathbf{\Lambda}_{\mathbb{R}} = \mathbf{M}_v$. Finally, by applying $\boldsymbol{\mathcal{P}}$ to the system $\mathcal{S}ys_N\left(\mathbf{U}_N, u_{1,\mathbb{R}}, u_{2,\mathbb{R}}\right)$, we obtain $\widehat{\mathbf{M}}_N^v = \boldsymbol{\mathcal{P}}\widehat{\mathbf{M}}_N^u\boldsymbol{\mathcal{P}}^T$ and $\hat{\mathbf{t}}_N^v = \boldsymbol{\mathcal{P}}\hat{\mathbf{t}}_N^u$. Moreover, since $\mathbf{v}_n = \boldsymbol{\mathcal{P}}\mathbf{u}_n$, $\forall n$, we have $\mathbf{M}_u = \mathbf{M}_v = \boldsymbol{\mathcal{P}}\mathbf{M}_u\boldsymbol{\mathcal{P}}^T$ and $\mathbf{t}_v = \boldsymbol{\mathcal{P}}\mathbf{t}_u$.

2) Link between asymptotic behaviors of $\left(\hat{\mathbf{t}}_N^u, \widehat{\mathbf{M}}_N^u\right)$, $\left(\hat{\mathbf{t}}_N^v, \widehat{\mathbf{M}}_N^v\right)$ *and* $\left(\hat{\mathbf{t}}_N^{\mathbb{R}}, \widehat{\mathbf{M}}_N^{\mathbb{R}}\right)$*:*

Lemma IV.1. $\hat{\mathbf{t}}_N^{\mathbb{R}}$ *and* $\hat{\mathbf{t}}_N^u$ *(respectively* $\widehat{\mathbf{M}}_N^{\mathbb{R}}$ *and* $\frac{1}{2}\left(\widehat{\mathbf{M}}_N^u + \widehat{\mathbf{M}}_N^v\right)$*) share the same asymptotic Gaussian law.*

Proof: See Appendix. ■

3) Asymptotic behavior of $\left(\hat{\mathbf{t}}_N, \widehat{\mathbf{M}}_N\right)$*:* Since $\text{vec}\left(\widehat{\mathbf{M}}_N^{\mathbb{R}}\right)$ and $\hat{\mathbf{t}}_N^{\mathbb{R}}$ have an asymptotic Gaussian distribution according to Lemma IV.1, and $\text{vec}\left(\widehat{\mathbf{M}}_N\right) = \left(\mathbf{g}_m^T \otimes \mathbf{g}_m^H\right)\text{vec}\left(\widehat{\mathbf{M}}_N^{\mathbb{R}}\right)$ and $\hat{\mathbf{t}}_N = \mathbf{g}_m^H\hat{\mathbf{t}}_N^{\mathbb{R}}$ with $\mathbf{g}_m = \left(\mathbf{I}_m, -j\mathbf{I}_m\right)^T$. Consequently, $\hat{\mathbf{t}}_N$ and $\text{vec}\left(\widehat{\mathbf{M}}_N\right)$ have a non-circular complex Gaussian distribution [27]. Additionally, using the same approach as in [25], we obtain

$$\sqrt{N}\text{vec}\left(\widehat{\mathbf{M}}_N - \mathbf{M}_e\right) \xrightarrow{d} \mathbf{b} \sim \mathcal{GCN}\left(\mathbf{0}, \mathbf{\Sigma}_M, \mathbf{\Omega}_M\right) \tag{13}$$

Regarding the location estimate, we have

$$\sqrt{N}\left(\hat{\mathbf{t}}_N - \mathbf{t}_e\right) \xrightarrow{d} \mathcal{GCN}\left(\mathbf{0}, \mathbf{\Sigma}_t, \mathbf{\Omega}_t\right), \text{ where} \tag{14}$$

$$\mathbf{\Sigma}_t = \mathbf{g}_m^H N\mathbb{E}\left[\left(\hat{\mathbf{t}}_N^u - \mathbf{t}_u\right)\left(\hat{\mathbf{t}}_N^u - \mathbf{t}_u\right)^T\right]\mathbf{g}_m = \frac{\alpha}{\beta^2}\mathbf{g}_m^H\mathbf{M}_{\mathbb{R}}\mathbf{g}_m = \frac{\alpha}{\beta^2}\mathbf{M}_e$$

in which

$$\begin{cases} \alpha = (2m)^{-1}\mathbb{E}\left[\psi_{1,\mathbb{R}}^2(\sqrt{\sigma}|\mathbf{x}|)\right] = m^{-1}\mathbb{E}\left[\psi_1^2(\sqrt{\sigma}|\boldsymbol{\zeta}|)\right] \\ \beta = \mathbb{E}\left[\left(1-(2m)^{-1}\right)u_1(\sqrt{\sigma}|\boldsymbol{\zeta}|) + (2m)^{-1}\psi_1^{'}\left(\sqrt{\sigma}|\boldsymbol{\zeta}|\right)\right] \end{cases}$$

with $\mathbf{x} \sim \mathcal{ES}_p\left(\mathbf{0}_{2m}, \mathbf{I}_{2m}, g_{\mathbf{z}}\right)$ and $\boldsymbol{\zeta} \sim \mathbb{C}\mathcal{ES}_m\left(\mathbf{0}_m, \mathbf{I}_m, g_{\mathbf{z}}\right)$ hence $|\boldsymbol{\zeta}| \stackrel{d}{=} 2^{-1/2}|\mathbf{x}|$. Furthermore, we have the relations $\psi_{1,\mathbb{R}}(s) = \sqrt{2}\psi_1\left(2^{-1/2}s\right)$ and $\psi_{1,\mathbb{R}}^{'}(s) = \psi_1^{'}\left(2^{-1/2}s\right)$. Moreover, we have

$$\mathbf{\Omega}_t = \mathbf{g}_m^H N\mathbb{E}\left[\left(\hat{\mathbf{t}}_N^{\mathbb{R}} - \mathbf{t}_{\mathbb{R}}\right)\left(\hat{\mathbf{t}}_N^{\mathbb{R}} - \mathbf{t}_{\mathbb{R}}\right)^T\right]\mathbf{g}_m^* = \frac{\alpha}{\beta^2}\mathbf{g}_m^H\mathbf{M}_{\mathbb{R}}\mathbf{g}_m^* = \mathbf{0}.$$

Thus, we prove that $\sqrt{N}\left(\hat{\mathbf{t}}_N - \mathbf{t}_e\right) \xrightarrow{d} \mathbf{a} \sim \mathbb{C}\mathcal{N}\left(\mathbf{0}, \mathbf{\Sigma}_t\right)$. Applying Lemma IV.1, we have $\boldsymbol{\omega} \perp \boldsymbol{\chi}$ and consequently,

$$\begin{cases} \mathbf{a} \stackrel{d}{=} \mathbf{g}_m^H\mathbf{M}_{\mathbb{R}}^{1/2}\mathbf{D}^{-1}\boldsymbol{\chi} \\ \mathbf{b} \stackrel{d}{=} \frac{1}{2}\left(\mathbf{g}_m^T \otimes \mathbf{g}_m^H\right)\left(\mathbf{I}_{p^2} + \left(\boldsymbol{\mathcal{P}} \otimes \boldsymbol{\mathcal{P}}\right)\right)\left(\mathbf{M}_{\mathbb{R}}^{1/2} \otimes \mathbf{M}_{\mathbb{R}}^{1/2}\right)\mathbf{A}^{-1}\boldsymbol{\omega} \end{cases},$$

thus we prove that $\mathbf{a} \perp \mathbf{b}$.

V. Simulations

In order to illustrate our theoretical results, some simulations results are presented. Two scenarios have been considered for the simulations. For $m = 3$, the true location parameter is $\mathbf{t}_e = (1 + 0.5i, 2 + i, 3 + 1.5i)^T$ and the true scatter matrix is $\mathbf{\Lambda} = \mathbf{I}_m$, due to the affine equivariance propertie of the M-estimators, there is no loss of generality.

- Case 1 : the data are generated under a t-distribution with $d = 4$ degrees of freedom [34].
- Case 2 : the data are generated under a K-distribution with shape parameter $\nu = 4$ and scale parameter $\theta = 1/\nu$ [26].

The complex joint M-estimators is obtained with $u(s) \triangleq u_2(s) = \dfrac{d+m}{d+s} = u_1(\sqrt{s})$ and the reweighting algorithm of [30], whose convergence is established.

The first case coincides with the MLE unlike the case 2, which is a general complex joint M-estimator.

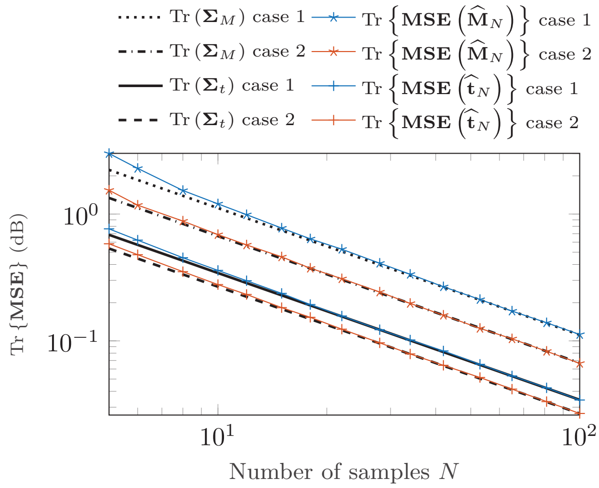

Fig. 1: Second order moment simulations

In Fig. 1, we plot the trace of the Mean Squared Error of the estimates of the location and the scatter matrix as well as the trace of the theoretical asymptotic covariance matrices $\mathbf{\Sigma}_t$ and $\mathbf{\Sigma}_M$. The results are validated since the drawn quantities are identical when $N \to \infty$. Moreover these quantities tend asymptotically to zero, which illustrate the consistency.

VI. Conclusion

In this letter, we established the asymptotic performance of the joint M-estimators for the complex multivariate location and scatter matrix. This statistical study highlights a better understanding on the performance of the M-estimators of the complex multivariate location and scatter matrix. Again, the obtained results may be used for conducting a performance analysis of adaptive processes involving non-zero mean observations.

TABLE I: Equations for the proof of Lemma IV.1

$$\mathbf{I}_p + \Delta\mathbf{M}_u = \frac{1}{N}\sum_{n=1}^{N} a_n\left(\mathbf{k}_n\mathbf{k}_n^T - \mathbf{k}_n\Delta\mathbf{t}_u^T\mathbf{M}_u^{-1/2} - \mathbf{M}_u^{-1/2}\Delta\mathbf{t}_u\mathbf{k}_n^T\right) + b_n\left[\mathbf{k}_n^T\Delta\mathbf{M}_u\mathbf{k}_n + 2\mathbf{k}_n^T\mathbf{M}_u^{-1/2}\Delta\mathbf{t}_u\right]\mathbf{k}_n\mathbf{k}_n^T \tag{15}$$

$$\mathbf{0} = \frac{1}{N}\sum_{n=1}^{N} c_n\left(\mathbf{k}_n - \mathbf{M}_u^{-1/2}\Delta\mathbf{t}_u\right) + \frac{1}{N}\sum_{n=1}^{N} d_n\left[\frac{\mathbf{k}_n^T\Delta\mathbf{M}_u\mathbf{k}_n}{2|\mathbf{k}_n|} + \frac{\mathbf{k}_n^T\mathbf{M}_u^{-1/2}\Delta\mathbf{t}_u}{|\mathbf{k}_n|}\right]\mathbf{k}_n \tag{16}$$

$$\mathbf{A}_N = \mathbf{I}_{p^2} - \frac{1}{N}\sum_{n=1}^{N} b_n\left(\mathbf{k}_n\otimes\mathbf{k}_n\right)\left(\mathbf{k}_n\otimes\mathbf{k}_n\right)^T \xrightarrow{\mathbb{P}} \mathbf{A} = \mathbf{I}_{p^2} + \frac{1}{p(p+2)}\mathbb{E}\left[|\mathbf{k}_1|^4 u_{2,\mathbb{R}}^{'}\left(|\mathbf{k}_1|^2\right)\right]\left(\mathbf{I}_{p^2} + \mathbf{K}_p + \mathrm{vec}\left(\mathbf{I}_p\right)\mathrm{vec}\left(\mathbf{I}_p\right)^T\right) \tag{17}$$

$$\mathbf{B}_N = \frac{1}{N}\sum_{n=1}^{N}\left(\mathbf{k}_n\otimes\mathbf{I}_p\right)\left(a_n\mathbf{I}_p - b_n\mathbf{k}_n\mathbf{k}_n^T\right) \xrightarrow{\mathbb{P}} \mathbb{E}\left[|\mathbf{k}_1|u_{2,\mathbb{R}}\left(|\mathbf{k}_1|^2\right)\right]\mathbb{E}\left[\left(\boldsymbol{\kappa}\otimes\mathbf{I}_p\right)\right] + \mathbb{E}\left[|\mathbf{k}_1|^3 u_{2,\mathbb{R}}^{'}\left(|\mathbf{k}_1|^2\right)\right]\mathbb{E}\left[\left(\boldsymbol{\kappa}\otimes\mathbf{I}_p\right)\boldsymbol{\kappa}\boldsymbol{\kappa}^T\right] = \mathbf{0} \tag{18}$$

$$\mathbf{C}_N = -\frac{1}{N}\sum_{n=1}^{N} d_n\frac{\mathbf{k}_n\mathbf{k}_n^T}{2|\mathbf{k}_n|}\left(\mathbf{I}_p\otimes\mathbf{k}_n\right)^T \xrightarrow{\mathbb{P}} \mathbb{E}\left[u_{1,\mathbb{R}}^{'}\left(|\mathbf{k}_1|\right)\frac{\mathbf{k}_1\mathbf{k}_1^T}{2|\mathbf{k}_1|}\left(\mathbf{I}_p\otimes\mathbf{k}_1\right)^T\right] = \frac{1}{2}\mathbb{E}\left[|\mathbf{k}_1|^2 u_{1,\mathbb{R}}^{'}\left(|\mathbf{k}_1|\right)\right]\mathbb{E}\left[\boldsymbol{\kappa}\boldsymbol{\kappa}^T\left(\mathbf{I}_p\otimes\boldsymbol{\kappa}\right)\right] = \mathbf{0} \tag{19}$$

$$\mathbf{D}_N = \frac{1}{N}\sum_{n=1}^{N} c_n\mathbf{I}_p - d_n\frac{\mathbf{k}_n\mathbf{k}_n^T}{|\mathbf{k}_n|} \xrightarrow{\mathbb{P}} \mathbf{D} = \left(\mathbb{E}\left[u_{1,\mathbb{R}}\left(|\mathbf{k}_1|\right)\right] + \frac{1}{p}\mathbb{E}\left[|\mathbf{k}_1|u_{1,\mathbb{R}}^{'}\left(|\mathbf{k}_1|\right)\right]\right)\mathbf{I}_p \tag{20}$$

$$\left(\mathbf{I}_{p^2} + \left(\boldsymbol{\mathcal{P}}\otimes\boldsymbol{\mathcal{P}}\right)\right)\boldsymbol{\omega}_N/\sqrt{N} = \left(\mathbf{A}_N + \left(\boldsymbol{\mathcal{P}}\otimes\boldsymbol{\mathcal{P}}\right)\mathbf{A}_N\left(\boldsymbol{\mathcal{P}}\otimes\boldsymbol{\mathcal{P}}\right)^T\right)\mathrm{vec}\left(\Delta\mathbf{M}_{\mathbb{R}}\right) + \left(\mathbf{I}_{p^2} + \mathbf{K}_p\right)\left(\mathbf{I}_{p^2} + \left(\boldsymbol{\mathcal{P}}\otimes\boldsymbol{\mathcal{P}}\right)\right)\mathbf{B}_N\mathbf{M}_{\mathbb{R}}^{-1/2}\Delta\mathbf{t}_{\mathbb{R}} \tag{21}$$

$$\mathbb{E}\left[\boldsymbol{\omega}\boldsymbol{\chi}^T\right] = \mathbb{E}\left[|\mathbf{k}_1|^3 u_{2,\mathbb{R}}\left(|\mathbf{k}_1|^2\right)u_{1,\mathbb{R}}\left(|\mathbf{k}_1|\right)\right]\mathbb{E}\left[\left(\boldsymbol{\kappa}\otimes\boldsymbol{\kappa}\right)\boldsymbol{\kappa}^T\right] - \mathbb{E}\left[|\mathbf{k}_1|u_{2,\mathbb{R}}\left(|\mathbf{k}_1|^2\right)u_{1,\mathbb{R}}\left(|\mathbf{k}_1|\right)\right]\mathbb{E}\left[\mathrm{vec}\left(\mathbf{I}_p\right)\boldsymbol{\kappa}^T\right] = \mathbf{0} \tag{22}$$

Appendix
Proof of Lemma IV.1

A. Asymptotic behavior of $\left(\hat{\mathbf{t}}_N^u, \widehat{\mathbf{M}}_N^u\right)$ and $\left(\hat{\mathbf{t}}_N^v, \widehat{\mathbf{M}}_N^v\right)$

First of all, let us define $\widehat{\mathbf{W}}_u = \mathbf{M}_u^{-1/2}\widehat{\mathbf{M}}_N^u\mathbf{M}_u^{-1/2}$ and $\mathbf{k}_n = \mathbf{M}_u^{-1/2}\left(\mathbf{u}_n - \mathbf{t}_u\right)$. Since $\left(\hat{\mathbf{t}}_N^u, \widehat{\mathbf{M}}_N^u\right) \xrightarrow{\mathbb{P}} \left(\mathbf{t}_u, \mathbf{M}_u\right)$, we can write for $N\to\infty$, $\widehat{\mathbf{W}}_u = \mathbf{I}_p + \Delta\mathbf{M}_u$ and $\hat{\mathbf{t}}_N^u = \mathbf{t}_u + \Delta\mathbf{t}_u$. Let us note $a_n = u_{2,\mathbb{R}}\left(|\mathbf{k}_n|^2\right)$, $b_n = -u_{2,\mathbb{R}}^{'}\left(|\mathbf{k}_n|^2\right)$, $c_n = u_{1,\mathbb{R}}\left(|\mathbf{k}_n|\right)$ and $d_n = -u_{1,\mathbb{R}}^{'}\left(|\mathbf{k}_n|\right)$ and use first order expansions for N sufficiently large, then we obtain (15) and (16) from $\mathcal{S}ys_N\left(\mathbf{U}_N, u_{1,\mathbb{R}}, u_{2,\mathbb{R}}\right)$. By vectorizing (15) and after some calculus, we obtain

$$\begin{cases} \boldsymbol{\omega}_N = \mathbf{A}_N\sqrt{N}\mathrm{vec}\left(\Delta\mathbf{M}_u\right) + \left(\mathbf{I}_{p^2} + \mathbf{K}_p\right)\mathbf{B}_N\sqrt{N}\mathbf{M}_u^{-1/2}\Delta\mathbf{t}_u \\ \boldsymbol{\chi}_N = \mathbf{C}_N\sqrt{N}\mathrm{vec}\left(\Delta\mathbf{M}_u\right) + \mathbf{D}_N\sqrt{N}\mathbf{M}_u^{-1/2}\Delta\mathbf{t}_u \end{cases} \tag{23}$$

where $\sqrt{N}\boldsymbol{\omega}_N = \sum_{n=1}^{N} a_n\left(\mathbf{k}_n\otimes\mathbf{k}_n\right) - \mathrm{vec}\left(\mathbf{I}_p\right)$ and $\sqrt{N}\boldsymbol{\chi}_N = \sum_{n=1}^{N} c_n\mathbf{k}_n$.

Remark. *Note that* $\mathbf{k}_n \sim \mathcal{ES}_p\left(\mathbf{0}, \sigma_{\mathbb{R}}\mathbf{I}_p, g_z\right)$*. Let be* $\boldsymbol{\kappa} = \frac{\mathbf{k}_1}{|\mathbf{k}_1|}$*, thus we have* $\boldsymbol{\kappa} \perp |\mathbf{k}_1|$ *and* $\mathbb{E}\left[\boldsymbol{\kappa}\right] = \mathbf{0}$*,* $\mathbb{E}\left[\boldsymbol{\kappa}\boldsymbol{\kappa}^T\right] = \frac{\mathbf{I}_p}{p}$*, all 3rd-order moments vanish and the only non-vanishing 4th-order moments are* $\mathbb{E}\left[\kappa_i^4\right] = \frac{3}{p(p+2)}$ *and* $\mathbb{E}\left[\kappa_i^2\kappa_j^2\right]^{-1} = p(p+2)$ *for* $i \neq j$ *where* $\boldsymbol{\kappa} = \left[\kappa_1, \ldots, \kappa_p\right]^T$*.*

The SLLN yields to (17)–(20). Furthermore, since $N^{-1}\sum_{n=1}^{N} a_n\left(\mathbf{k}_n\otimes\mathbf{k}_n\right) \xrightarrow{\mathbb{P}} \mathrm{vec}\left(\mathbf{I}_p\right)$ and $N^{-1}\sum_{n=1}^{N} c_n\mathbf{k}_n \xrightarrow{\mathbb{P}} \mathbf{0}_p$, it yields from the central limit theorem that $\boldsymbol{\omega}_N \xrightarrow{d} \boldsymbol{\omega}$ and $\boldsymbol{\chi}_N \xrightarrow{d} \boldsymbol{\chi}$ with $\boldsymbol{\omega}$ and $\boldsymbol{\chi}$ zero-mean Gaussian distributed. Applying Slutsky's lemma [32], it comes

$$\sqrt{N}\left(\hat{\mathbf{t}}_N^u - \mathbf{t}_u\right) \stackrel{d}{=} \sqrt{N}\Delta\mathbf{t}_u \xrightarrow{d} \mathbf{M}_u^{1/2}\mathbf{D}^{-1}\boldsymbol{\chi}$$
$$\sqrt{N}\mathrm{vec}\left(\widehat{\mathbf{M}}_N^u - \mathbf{M}_u\right) \xrightarrow{d} \left(\mathbf{M}_u^{1/2}\otimes\mathbf{M}_u^{1/2}\right)\mathbf{A}^{-1}\boldsymbol{\omega}$$

In the same way, we obtain

$$\sqrt{N}\mathrm{vec}\left(\widehat{\mathbf{M}}_N^v - \mathbf{M}_v\right) \xrightarrow{d} \left(\boldsymbol{\mathcal{P}}\otimes\boldsymbol{\mathcal{P}}\right)\left(\mathbf{M}_u^{1/2}\otimes\mathbf{M}_u^{1/2}\right)\mathbf{A}^{-1}\boldsymbol{\omega}$$
$$\sqrt{N}\left(\hat{\mathbf{t}}_N^v - \mathbf{t}_v\right) \xrightarrow{d} \boldsymbol{\mathcal{P}}\mathbf{M}_u^{1/2}\mathbf{D}^{-1}\boldsymbol{\chi}$$

B. Asymptotic behavior of $\left(\hat{\mathbf{t}}_N^{\mathbb{R}}, \widehat{\mathbf{M}}_N^{\mathbb{R}}\right)$

With the results of Theorem III.1, the continuous mapping theorem implies

$\left(\hat{\mathbf{t}}_N^{\mathbb{R}}, \widehat{\mathbf{M}}_N^{\mathbb{R}}\right) = \left(h\left(\hat{\mathbf{t}}_N\right), f\left(\widehat{\mathbf{M}}_N\right)\right) \xrightarrow{\mathbb{P}} \left(h\left(\mathbf{t}_e\right), f\left(\mathbf{M}_e\right)\right) = \left(\mathbf{t}_{\mathbb{R}}, \mathbf{M}_{\mathbb{R}}\right)$ and $\left(h\left(\mathbf{t}_e\right), f\left(\mathbf{M}_e\right)\right) = \left(h\left(\mathbf{t}_e\right), \sigma^{-1}f\left(\boldsymbol{\Lambda}\right)\right) = \left(h\left(\mathbf{t}_e\right), \sigma^{-1}\boldsymbol{\Lambda}_{\mathbb{R}}\right) = \left(\mathbf{t}_u, \sigma_{\mathbb{R}}^{-1}\boldsymbol{\Lambda}_{\mathbb{R}}\right)$

Let us define $\widehat{\mathbf{W}}_{\mathbb{R}} = \mathbf{M}_{\mathbb{R}}^{-1/2}\widehat{\mathbf{M}}_N^{\mathbb{R}}\mathbf{M}_{\mathbb{R}}^{-1/2}$. Since $\mathbf{M}_{\mathbb{R}} = \mathbf{M}_u$ and $\mathbf{t}_{\mathbb{R}} = \mathbf{t}_u$, $\mathbf{k}_n$ becomes $\mathbf{k}_n = \mathbf{M}_{\mathbb{R}}^{-1/2}\left(\mathbf{u}_n - \mathbf{t}_{\mathbb{R}}\right)$ and satisfies $\boldsymbol{\mathcal{P}}\mathbf{k}_n = \mathbf{M}_{\mathbb{R}}^{-1/2}\left(\mathbf{v}_n - \boldsymbol{\mathcal{P}}\mathbf{t}_{\mathbb{R}}\right)$. For $N\to\infty$, we can write $\widehat{\mathbf{W}}_{\mathbb{R}} = \mathbf{I}_p + \Delta\mathbf{M}_{\mathbb{R}}$ and $\hat{\mathbf{t}}_N^{\mathbb{R}} = \mathbf{t}_{\mathbb{R}} + \Delta\mathbf{t}_{\mathbb{R}}$. As previously, from (10) we obtain (21). Since $\left(\mathbf{A}_N + \left(\boldsymbol{\mathcal{P}}\otimes\boldsymbol{\mathcal{P}}\right)\mathbf{A}_N\left(\boldsymbol{\mathcal{P}}\otimes\boldsymbol{\mathcal{P}}\right)^T\right) \xrightarrow{\mathbb{P}} 2\mathbf{A}$, the Slutsky's lemma leads to

$$\sqrt{N}\mathrm{vec}\left(\widehat{\mathbf{M}}_N^{\mathbb{R}} - \mathbf{M}_{\mathbb{R}}\right) \xrightarrow{d} \frac{1}{2}\left(\mathbf{I}_{p^2} + \boldsymbol{\mathcal{P}}\otimes\boldsymbol{\mathcal{P}}\right)\left(\mathbf{M}_u^{1/2}\otimes\mathbf{M}_u^{1/2}\right)\mathbf{A}^{-1}\boldsymbol{\omega}$$

Similarly, from (11) we obtain

$$\boldsymbol{\chi}_N = \mathbf{C}_N\sqrt{N}\mathrm{vec}\left(\Delta\mathbf{M}_{\mathbb{R}}\right) + \mathbf{D}_N\sqrt{N}\mathbf{M}_{\mathbb{R}}^{-1/2}\Delta\mathbf{t}_{\mathbb{R}}$$

and thus $\sqrt{N}\left(\hat{\mathbf{t}}_N^{\mathbb{R}} - \mathbf{t}_{\mathbb{R}}\right) \xrightarrow{d} \mathbf{M}_{\mathbb{R}}^{1/2}\mathbf{D}^{-1}\boldsymbol{\chi}$, which means that $\hat{\mathbf{t}}_N^{\mathbb{R}}$ and $\hat{\mathbf{t}}_N^u$ have the same asymptotic distribution.

Lastly, we also introduce $\boldsymbol{\xi}_n = \begin{bmatrix} a_n\left(\mathbf{k}_n\otimes\mathbf{k}_n\right) - \mathrm{vec}\left(\mathbf{I}_p\right) \\ c_n\mathbf{k}_n \end{bmatrix}$, which are zero mean and i.i.d., then

$$\begin{pmatrix}\boldsymbol{\omega}_N \\ \boldsymbol{\chi}_N\end{pmatrix} = \frac{1}{\sqrt{N}}\sum_{n=1}^{N}\boldsymbol{\xi}_n \xrightarrow{d} \boldsymbol{\xi} = \begin{pmatrix}\boldsymbol{\omega} \\ \boldsymbol{\chi}\end{pmatrix} \sim \mathcal{N}\left(\mathbf{0}, \mathrm{Var}\left(\boldsymbol{\xi}_1\right)\right)$$

Furthermore, since we have (22), we obtain $\boldsymbol{\omega} \perp \boldsymbol{\chi}$.

References

[1] J. Frontera-Pons, J.-P. Ovarlez, and F. Pascal, "Robust ANMF detection in noncentered impulsive background," *IEEE Signal Processing Letters*, vol. 24, no. 12, pp. 1891–1895, Dec. 2017.

[2] J. A. Fessler, "Mean and variance of implicitly defined biased estimators (such as penalized maximum likelihood): applications to tomography," *IEEE Transactions on Image Processing*, vol. 5, no. 3, pp. 493–506, Mar. 1996.

[3] A. M. Zoubir, V. Koivunen, Y. Chakhchoukh, and M. Muma, "Robust estimation in signal processing: A tutorial-style treatment of fundamental concepts," *IEEE Signal Processing Magazine*, vol. 29, no. 4, pp. 61–80, Jul. 2012.

[4] V. Ollier, M. N. E. Korso, R. Boyer, P. Larzabal, and M. Pesavento, "Robust calibration of radio interferometers in non-gaussian environment," *IEEE Transactions on Signal Processing*, vol. 65, no. 21, pp. 5649–5660, Nov. 2017.

[5] X. Zhang, M. N. E. Korso, and M. Pesavento, "MIMO radar target localization and performance evaluation under SIRP clutter," *Elsevier Signal Processing*, vol. 130, pp. 217–232, Jan. 2017.

[6] M. Hubert, P. Rousseeuw, D. Vanpaemel, and T. Verdonck, "The detS and detMM estimators for multivariate location and scatter," *Computational Statistics and Data Analysis*, vol. 81, pp. 64–75, 2015.

[7] R. A. Maronna and V. J. Yohai, "Robust and efficient estimation of multivariate scatter and location," *Computational Statistics and Data Analysis*, vol. 109, pp. 64–75, 2017.

[8] R. A. Maronna, "Robust M-estimators of multivariate location and scatter," *The Annals of Statistics*, vol. 4, no. 1, pp. 51–67, Jan. 1976.

[9] P. J. Rousseeuw, "Multivariate estimation with high breakdown point," *Mathematical statistics and applications*, vol. 8, pp. 283–297, 1985.

[10] ——, "Least median of squares regression," *Journal of the American Statistical Association*, vol. 79, no. 388, pp. 871–880, 1984.

[11] M. Hubert and M. Debruyne, "Minimum covariance determinant," *Wiley Interdisciplinary Reviews: Computational Statistics*, vol. 2, no. 1, pp. 36–43, Dec. 2009.

[12] W. A. Stahel, "Breakdown of covariance estimators," E.T.H. Zürich, Tech. Rep., 1981.

[13] D. L. Donoho, "Breakdown properties of multivariate location estimators," Ph.D. dissertation, Harvard University, 1982.

[14] P. Rousseeuw and A. Leroy, *Robust Regression and Outlier Detection*, ser. Wiley Series in Probability and Statistics. Wiley, 1987.

[15] H. P. Lopuhaa, "On the relation between S-estimators and M-estimators of multivariate location and covariance," *The Annals of Statistics*, vol. 17, no. 4, pp. 1662–1683, 1989.

[16] V. Yohai, "High breakdown-point and high efficiency robust estimates for regression," *The Annals of Statistics*, vol. 15, no. 2, pp. 642–656, Jun. 1987.

[17] K. S. Tatsuoka and D. E. Tyler, "On the uniqueness of s-functionals and m-functionals under nonelliptical distributions," *The Annals of Statistics*, vol. 28, no. 4, pp. 1219–1243, 2000.

[18] G. Frahm, "Generalized elliptical distributions: Theory and applications," Ph.D. dissertation, Universität zu Köln, 2004.

[19] K. Fang, S. Kotz, and K. Wang NG, *Symmetric Multivariate and Related Distributions*. Chapman and Hall, 1990.

[20] D. E. Tyler, "Radial estimates and the test for sphericity," *Biometrika*, vol. 69, no. 2, pp. 429–436, 1982.

[21] M. Bilodeau and D. Brenner, *Theory of multivariate statistics*, 1st ed. Springer Science & Business Media, 2008.

[22] O. Arslan, "Convergence behavior of an iterative reweighting algorithm to compute multivariate m-estimates for location and scatter," *Journal of Statistical Planning and Inference*, vol. 1, no. 118, pp. 115–128, 2004.

[23] E. Ollila and V. Koivunen, "Robust antenna array processing using M-estimators of pseudo-covariance," in *Proc. of IEEE International Symposium on Personal, Indoor and Mobile Radio Communications (PIMRC)*, vol. 3, Sep. 2003, pp. 2659–2663.

[24] M. Greco and F. Gini, "Cramér-Rao lower bounds on covariance matrix estimation for complex elliptically symmetric distributions," *IEEE Transactions on Signal Processing*, vol. 61, no. 24, pp. 6401–6409, 2013.

[25] M. Mahot, F. Pascal, P. Forster, and J.-P. Ovarlez, "Asymptotic properties of robust complex covariance matrix estimates," *IEEE Transactions on Signal Processing*, vol. 61, no. 13, pp. 3348–3356, Jul. 2013.

[26] E. Ollila, D. E. Tyler, V. Koivunen, and H. V. Poor, "Complex elliptically symmetric distributions: Survey, new results and applications," *IEEE Transactions on Signal Processing*, vol. 60, no. 11, pp. 5597–5625, Nov. 2012.

[27] A. Van den Bos, "The multivariate complex normal distribution - A generalization," *IEEE Transactions on Information Theory*, vol. 41, no. 2, pp. 537–539, Mar. 1995.

[28] J.-P. Delmas, "Performance bounds and statistical analysis of doa estimation," in *Array and statistical signal processing*, ser. Academic Press Library in Signal Processing. Elsevier, Dec. 2014, vol. 3, ch. 16, pp. 719–764.

[29] D. Kelker, "Distribution theory of spherical distributions and a location-scale parameter generalization," *Sankhya: The Indian Journal of Statistics, Series A*, vol. 4, no. 32, pp. 419–430, 1970.

[30] J. T. Kent and D. E. Tyler, "Redescending M-estimates of multivariate location and scatter," *The Annals of Statistics*, vol. 19, no. 4, pp. 2102–2119, Dec. 1991.

[31] A. W. Van der Vaart, *Asymptotic Statistics (Cambridge Series in Statistical and Probabilistic Mathematics)*. Cambridge University Press, Jun. 2000, vol. 3.

[32] R. Maronna, R. Martin, and V. Yohai, *Robust statistics, Theory and Methods*. John Wiley & Sons, Chichester, 2006, vol. 1.

[33] J. R. Magnus and H. Neudecker, "The commutation matrix: some properties and applications," *The Annals of Statistics*, vol. 7, no. 2, pp. 381–394, Mar. 1979.

[34] O. Besson and Y. I. Abramovich, "On the fisher information matrix for multivariate elliptically contoured distributions," *IEEE Signal Processing Letters*, vol. 20, no. 11, pp. 1130–1133, Nov. 2013.